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14. ABSTRACT
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ON COMPUTING THE $\mathbf{H}_\infty$ NORM OF A TRANSFER MATRIX

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Abstract

We present a simple bisection algorithm to compute the $\mathbf{H}_\infty$ norm of a transfer matrix. The bisection method is far more efficient than algorithms which involve a search over frequencies, and moreover can compute the $\mathbf{H}_\infty$ norm with guaranteed accuracy.

1 Preliminaries

Throughout this paper A, B, C, D will be real matrices of sizes $n \times n$, $n \times m$, $p \times n$, and $p \times m$, respectively. We refer to the linear dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

as the *system* $\{A, B, C, D\}$. We refer to $H(s) = C(sI - A)^{-1}B + D$ as the *transfer matrix* of the system $\{A, B, C, D\}$.

A is *stable* means that all eigenvalues of A have negative real part. If A is stable, we define the $\mathbf{H}_\infty$ norm of the transfer matrix $H(s)$ to be

$$\begin{aligned} \|H\|_\infty &= \sup_{\mathrm{Re}\, s > 0} \sigma_{max}(H(s)) \\ &= \sup_{\omega \in \mathbf{R}} \sigma_{max}(H(j\omega)) \end{aligned} \tag{2}$$

where $\sigma_{max}(\cdot)$ denotes the maximum singular value of a matrix, that is, $\sigma_{max}(F) = \lambda_{max}^{1/2}(F^*F)$. The $\mathbf{H}_\infty$ norm of a transfer matrix arises often in control theory. An important interpretation of $\|H\|_\infty$ is as the $\mathbf{L}_2$ or RMS gain of the system (1) (see e.g. [2]).

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$\|H\|_\infty$ is usually 'computed' by searching for the maximum of $\sigma_{max}(H(j\omega))$ over $\omega \in \mathbf{R}$. Obvious problems associated with such a method are (a) determining the range and spacing of the frequencies to be checked, and (b) the large number of computations involved (a singular value decomposition (SVD) is often performed at each frequency point). The problem (a) is particularly evident when A has lightly damped eigenvalues.

We propose instead a bisection method inspired by Byers' bisection method for measuring the distance of a stable matrix to the unstable matrices [3]. The bisection method not only involves less computation, but has the advantage of computing $\|H\|_\infty$ with a guaranteed accuracy.

2 SVs of H via a Hamiltonian

We start by establishing a connection between the singular values of the transfer matrix and the imaginary eigenvalues of a certain Hamiltonian matrix. Let $\gamma > 0$, and not a singular value of D. Define

$$M_\gamma = \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix}$$

where $R = (D^TD - \gamma^2 I)$ and $S = (DD^T - \gamma^2 I)$. Note that M_γ is a *Hamiltonian* matrix.

The following theorem relates the singular values of $H(j\omega)$ and the imaginary eigenvalues of M_γ.

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Theorem 1 *Assume A has no imaginary eigenvalues, $\gamma > 0$ is not a singular value of D, and $\omega_0 \in \mathbf{R}$.*

Then, γ is a singular value of $H(j\omega_0) \iff (M_\gamma - j\omega_0 I)$ is singular.

Remark 1:

There are no observability, controllability, or stability conditions on the system $\{A, B, C, D\}$.

A simple consequence of Theorem 1 is

Theorem 2 *Let A be stable and $\gamma > \sigma_{max}(D)$. Then $\|H\|_\infty \geq \gamma \iff M_\gamma$ has imaginary eigenvalues (i.e. at least one).*

Remark 2:

The imaginary eigenvalues of $M_{\|H\|_\infty}$ are exactly the frequencies for which $\sigma_{max}(H(j\omega)) = \|H\|_\infty$.

Remark 3:

Theorem 2 is also readily derived via several methods, *e.g.*, quadratic optimal control [4] or spectral factorization [5].

3 Bisection Algorithm

Theorem 2 suggests a bisection algorithm for computing $\|H\|_\infty$. Let γ_{lb} and γ_{ub} be some lower and upper bounds, respectively, on $\|H\|_\infty$. For example, one could use the bounds derived by Enns and Glover,

$$\gamma_{lb} = \max\{\sigma_{max}(D),\ \sigma_{H1}\}$$

$$\gamma_{ub} = \sigma_{max}(D) + 2\sum_{i=1}^{n} \sigma_{Hi}$$

where σ_{Hi} are the Hankel singular values of the system $\{A, B, C, D\}$ [6, 7].

The bisection algorithm is as follows:

$\gamma_L := \gamma_{lb}$;
$\gamma_U := \gamma_{ub}$;
repeat {
  $\gamma := (\gamma_L + \gamma_H)/2$;
  Form M_γ;
  if M_γ *has no imag. eigenvalues,* $\gamma_H := \gamma$,
  else $\gamma_L = \gamma$; }
until { $\gamma_H - \gamma_L \leq 2\epsilon\ \gamma_L$ }.

Note that we always have $\gamma_L \leq \|H\|_\infty \leq \gamma_H$. On exit, $(\gamma_L + \gamma_H)/2$ is guaranteed to approximate $\|H\|_\infty$ within a relative accuracy of ϵ, i.e.,

$$|(\gamma_L + \gamma_H)/2 - \|H\|_\infty| \leq \epsilon \|H\|_\infty.$$

Remark 4:

Checking if M_γ has any imaginary eigenvalues can be done in a finite number of steps via a Sturm sequence test on the characteristic polynomial [8].

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